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LETTER TO THE EDITOR

A universal non-quasitriangular quantization of the Heisenberg group

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Abstract. A new universal R-matrix for the quantum Heisenberg algebra $\mathfrak{h}(1)_q$ is obtained by imposing the analyticity in the deformation parameter. Despite the non-quasitriangularity of this Hopf algebra, the quantum group induced from it coincides with the quasitriangular deformation already known.

Recall that a quasitriangular Hopf algebra [1] is a pair $(\mathcal{A}, \mathcal{R})$ where $\mathcal{A}$ is a Hopf algebra and $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ is invertible and verifies

$$\sigma \circ \Delta h = \mathcal{R}(\Delta h)\mathcal{R}^{-1} \qquad \forall h \in \mathcal{A} \tag{1a}$$

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \qquad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \tag{1b}$$

where, if $\mathcal{R} = \sum_i a_i \otimes b_i$, we denote $\mathcal{R}_{12} \equiv \sum_i a_i \otimes b_i \otimes 1$, $\mathcal{R}_{13} \equiv \sum_i a_i \otimes 1 \otimes b_i$, $\mathcal{R}_{23} \equiv \sum_i 1 \otimes a_i \otimes b_i$ and σ is the flip operator $\sigma(x \otimes y) = (y \otimes x)$. If $\mathcal{A}$ is a quasitriangular Hopf algebra, then $\mathcal{R}$ is called a 'universal' $\mathcal{R}$-matrix and satisfies the quantum Yang–Baxter equation (QYBE)

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{2}$$

Given a matrix representation $\rho : \mathcal{A} \to \mathrm{Mat}(n, \mathbb{C})$, the FRT approach [2] to quantum groups is based on an R-matrix taken as $R = (\rho \otimes \rho)(\mathcal{R})$. The matrix entries $t_{ij}(i, j = 1, \ldots, n)$ of a general 'quantum group' element T satisfy the commutation relations

$$RT_1T_2 = T_2T_1R \tag{3}$$

where $T = (t_{ij})$, $T_1 = T \otimes 1_n$ and $T_2 = 1_n \otimes T$. The Yang–Baxter (YB) condition ensures the associativity of the (non-commutative) algebra of functions on the quantum group $Fun(G_q)$ generated by the t_{ij}. It also provides a straightforward way to obtain $\mathcal{A}$ as the dual of $Fun(G_q)$ by using the $L^{\pm}$ matrices [1,3].

This letter shows the non-unicity of the R-matrix for the quantum Heisenberg algebra $\mathfrak{h}(1)_q$. There are two essentially different R-matrices, one of them quasitriangular [4,5] and the other, presented in this letter, non-quasitriangular. It is therefore natural to study whether

$Fun(G_q)$ can be derived with the aid of the FRT prescription using this non-quasitriangular R-matrix. We found this to be the case and, moreover, its associated quantum Heisenberg group coincides with the one given in [4, 6] (see [7] for a characterization of all possible quantizations of this group).

The essential point of the approach presented here is the assumption that the R-matrix and all the theory is analytical in the quantum parameter z (= $\log q$). Let us consider an R-matrix as a formal power series in z with coefficients in $U\mathfrak{h}(1) \otimes U\mathfrak{h}(1)$

$$R = e^{zR_1} + \sum_{i=1}^{\infty} z^i (e^{zR_{i+1}} - 1). \tag{4}$$

Provided the coproduct Δ is also expanded in terms of z, we can try to solve (1a) order by order in the deformation parameter and obtain the explicit form of the R_i components.

The *quantum Heisenberg algebra* $\mathfrak{h}(1)_q$ [4] *is defined by the commutation relations*

$$[A, A^\dagger] = (\sinh zH)/z \qquad [A, H] = 0 \qquad [A^\dagger, H] = 0 \tag{5}$$

which are consistent with the Hopf homomorphisms

$$\begin{aligned} &\Delta(H) = 1 \otimes H + H \otimes 1 \\ &\Delta(A^\dagger) = e^{-(z/2)H} \otimes A^\dagger + A^\dagger \otimes e^{(z/2)H} \\ &\Delta(A) = e^{-(z/2)H} \otimes A + A \otimes e^{(z/2)H} \end{aligned} \tag{6}$$

$$\epsilon(X) = 0 \qquad \gamma(X) = -e^{(z/2)H} X e^{-(z/2)H} = -X \qquad X \in \{A, A^\dagger, H\}. \tag{7}$$

If we assume the ansatz (4) for the R-matrix, the first order of (1a) leads to

$$[R_1, 1 \otimes X + X \otimes 1] = H \otimes X - X \otimes H. \tag{8}$$

Equation (8) has two different solutions:

$$R_1 = A \otimes A^\dagger - A^\dagger \otimes A \tag{9}$$

and

$$R_1' = H \otimes A^\dagger A + A^\dagger A \otimes H - 2A \otimes A^\dagger. \tag{10}$$

Note that the general R_1-matrix will be a linear combination of (9) and (10). Applying our procedure to the latter, we should obtain the quasitriangular R-matrix presented in [4, 5]. So, we restrict ourselves in the following to (9). Note that, as expected, R_1 is a classical r-matrix for the algebra $\mathfrak{h}(1)$ (it fulfils the modified classical YB equation). The study of the r-matrices for this algebra has been given in [6] where it is shown that (9) provides the (essentially unique) Poisson–Lie quantization of $H(1)$.

The second order leads to the condition

$$[R_2, 1 \otimes X + X \otimes 1] = 0 \tag{11}$$

for any generator X. This requirement is obviously fulfilled if $R_2 = 0$ and we shall adopt this solution.

The third order originates the following equation for R_3

$$[R_3, 1\otimes X + X\otimes 1] = \tfrac{1}{12}[R_1,[R_1,[R_1, 1\otimes X + X\otimes 1]]] - \tfrac{1}{8}[R_1, H^2\otimes X + X\otimes H^2] - \tfrac{1}{8}[R_1, H^3\otimes X + X\otimes H^3]. \tag{12}$$

A solution for (12) is given by

$$R_3 = -R_1(\tfrac{1}{12}H\otimes H + \tfrac{1}{8}(1\otimes H^2 + H^2\otimes 1)). \tag{13}$$

Note that R_3 exhibits a remarkable property: it depends only on R_1 and H. We put forward this dependence by supposing that R can be written as

$$R = \exp(zR_1 f(H,z)). \tag{14}$$

Under this assumption, the condition (1*a*) can be solved in general (from now on, we shall not write the z-dependence of the function f). Explicitly,

$$\begin{aligned} R\Delta X R^{-1} &= \exp(zR_1 f(H))\Delta X\exp(-zR_1 f(H)) \\ &= \Delta X + zf(H)[R_1,\Delta X] + (1/2!)z^2 f(H)^2[R_1,[R_1,\Delta X]] + \cdots \\ &\quad + (1/n!)z^n f(H)^n[R_1,[\ldots[R_1,\Delta X]]^{n)}\ldots] + \cdots . \end{aligned} \tag{15}$$

For $X\equiv A$ and $X\equiv A^\dagger$, we need to obtain the brackets in (15)

$$\begin{aligned} [R_1,\Delta X] &= [A\otimes A^\dagger - A^\dagger\otimes A, \mathrm{e}^{-(z/2)H}\otimes X + X\otimes \mathrm{e}^{(z/2)H}] \\ &= \frac{1}{z}\sinh(zH)\otimes X\mathrm{e}^{(z/2)H} - X\mathrm{e}^{-(z/2)H}\otimes\frac{1}{z}\sinh(zH) \end{aligned} \tag{16}$$

$$[R_1,[R_1,\Delta X]] = -\frac{1}{z^2}(\sinh zH\otimes\sinh zH)\Delta X \tag{17}$$

$$[R_1,[R_1,[R_1,\Delta X]]] = -\frac{1}{z^2}(\sinh zH\otimes\sinh zH)[R_1,\Delta X]. \tag{18}$$

In general, a recurrence method gives for $2n$ and $2n+1$ iterates, respectively

$$[R_1,[\ldots[R_1,\Delta X]]^{2n)}\ldots] = \frac{(-1)^n}{z^{2n}}(\sinh zH\otimes\sinh zH)^n\Delta X \tag{19}$$

$$[R_1,[\ldots[R_1,\Delta X]]^{2n+1)}\ldots] = \frac{(-1)^n}{z^{2n+1}}(\sinh zH\otimes\sinh zH)^n z[R_1,\Delta X]. \tag{20}$$

Now, expression (15) can be written as follows

$$\begin{aligned} \exp(zR_1 f(H))\Delta X\exp(-zR_1 f(H)) &= \sum_{l=0}^{\infty}\frac{(-1)^l}{(2l)!}[f(H)]^{2l}(\sinh zH\otimes\sinh zH)^l\Delta X \\ &\quad + \sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}[f(H)]^{2j+1}(\sinh zH\otimes\sinh zH)^j z[R_1,\Delta X] \\ &= \cos\left(\sqrt{\sinh zH\otimes\sinh zH}\,f(H)\right)\Delta X \\ &\quad + \frac{\sin\left(\sqrt{\sinh zH\otimes\sinh zH}\,f(H)\right)}{\sqrt{\sinh zH\otimes\sinh zH}}z[R_1,\Delta X] \qquad (21a) \\ &= \alpha(H)\Delta X + \beta(H)z[R_1,\Delta X] \qquad (21b) \end{aligned}$$

where $\alpha(H)$ and $\beta(H)$ are to be determined. We recall that (21) has to provide the same effect on ΔX as the flip operator σ. Explicitly, for $X \neq H$, we have

$$\begin{aligned}\sigma \circ \Delta X &= e^{(z/2)H} \otimes X + X \otimes e^{-(z/2)H} \\ &= (X \otimes 1)(1 \otimes e^{-(z/2)H}) + (e^{(z/2)H} \otimes 1)(1 \otimes X).\end{aligned} \tag{22}$$

If we impose (22) to coincide with (21*b*), we obtain the following system of equations for α and β:

$$\begin{aligned}&\alpha(H)(e^{-(z/2)H} \otimes 1) + \beta(H)(\sinh zH \otimes e^{(z/2)H}) = (e^{(z/2)H} \otimes 1) \\ &\alpha(H)(1 \otimes e^{(z/2)H}) - \beta(H)(e^{-(z/2)H} \otimes \sinh zH) = (1 \otimes e^{-(z/2)H}).\end{aligned} \tag{23}$$

These equations can be solved and lead to the solution

$$\begin{aligned}\alpha(H) &= \frac{1 \otimes 1 + e^{-zH} \otimes e^{zH}}{1 \otimes e^{zH} + e^{-zH} \otimes 1} = \frac{e^{(z/2)H} \otimes e^{-(z/2)H} + e^{-(z/2)H} \otimes e^{(z/2)H}}{\cosh((z/2)\Delta H)} \\ \beta(H) &= \frac{2}{e^{(z/2)H} \otimes e^{(z/2)H} + e^{-(z/2)H} \otimes e^{-(z/2)H}} = \frac{1}{\cosh((z/2)\Delta H)}.\end{aligned} \tag{24}$$

For the sake of consistency between (21) and (24), it can be easily checked that $\alpha(H)^2 + [\sinh(zH) \otimes \sinh(zH)]\beta(H)^2 = 1$. Finally, from these two equations, we deduce the final form of the function $f(H)$ which reads

$$f(H, z) = \frac{1}{\sqrt{\sinh zH \otimes \sinh zH}} \sin^{-1}\left(\frac{\sqrt{\sinh zH \otimes \sinh zH}}{\cosh((z/2)\Delta H)}\right) \tag{25}$$

or

$$f(H, z) = \frac{1}{\sqrt{\sinh zH \otimes \sinh zH}} \cos^{-1}\left(2\frac{e^{(z/2)H} \otimes e^{-(z/2)H} + e^{-(z/2)H} \otimes e^{(z/2)H}}{\cosh((z/2)\Delta H)}\right). \tag{26}$$

A power series expansion of $f(H, z)$ in terms of z leads to (4) with R_1 and R_3 given by (9) and (13), respectively. The R-matrix (14) so obtained does not fulfil the QYBE equation as can be easily shown by the second-order expansion series on z.

Let us now consider a three-dimensional representation of the Heisenberg group

$$T = \begin{pmatrix} 1 & \theta & a_1 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{27}$$

The corresponding quantum group is the (non-commutative) Hopf algebra of functions generated by the matrix entries t_{ij}. Their coproduct is induced from the group multiplication $T\dot{\otimes}T$

$$\begin{aligned}\Delta(\theta) &= 1 \otimes \theta + \theta \otimes 1 \\ \Delta(a_1) &= 1 \otimes a_1 + a_1 \otimes 1 + \theta \otimes a_2 \\ \Delta(a_2) &= 1 \otimes a_2 + a_2 \otimes 1.\end{aligned} \tag{28}$$

Co-unit and antipode are derived from the unit matrix and from T^{-1}, respectively (see [4]).

The fundamental representation of the quantum algebra (5) is

$$D(H) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad D(A^\dagger) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad D(A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{29}$$

If we realize the R-matrix (13) in terms of (28), we find that only the linear term in z is left (see also [8])

$$D(R) = 1_3 + z[D(A) \otimes D(A^\dagger) - D(A^\dagger) \otimes D(A)] = \begin{pmatrix} 1_3 & zD(A^\dagger) & 0 \\ 0 & 1_3 & -zD(A) \\ 0 & 0 & 1_3 \end{pmatrix}. \tag{30}$$

This R-matrix, together with (3) and (27), provides

$$[\theta, a_1] = z\theta \qquad [\theta, a_2] = 0 \qquad [a_1, a_2] = -za_2. \tag{31}$$

This algebra verifies the Jacobi identity and coincides (up to a change in the deformation parameter $z = w/2$) with the quantum Heisenberg group obtained from a quasitriangular Hopf algebra in [4]. In that paper, a number operator N was needed in order to obtain a universal $\mathcal{R}$-matrix in the same way as its classical counterpart n had been introduced to generate the same $\mathcal{R}$-matrix from the coboundary Poisson–Lie structure studied in [6]. On the other hand, the duality between relations (28) and (31) and the quantum Heisenberg algebra (5) has been given in [9]. Finally, note that the universality of R allows the FRT construction for any finite-dimensional representation of the Heisenberg group.

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